Notes on wavelet transforms

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1 Introduction

This is a report on the work carried out by the second author under the supervision of the first author during the period June 27 to August 27, 2004. The work consisted of learning the necessary mathematics for understanding some of the basic theory of wavelet transforms as presented by Ingrid Daubechies in [Daubechies 1988]. The text [Nievergelt 1999] was used as a guide. This text, while very useful, contains numerous errors. Most of these are either typographical or minor calculation errors. However, there also seem to be important gaps in some of the proofs. For the gaps we found, we indicate possible repairs. The repairs are drawn from the literature but with the results and proofs modified to make them accessible to a reader whose background includes calculus and linear algebra but not Lebesgue integration. Our review of the literature in the brief time we had for our work was limited to parts of the cited bibliography. Undoubtedly some of what we have done here is known to be achievable by simpler means. The results of some computational experimentation based on the theory of wavelet transforms are described in the last section.

We use the notation $\hat{f}$ for the fourier transform of f in addition to the notation $\mathcal{F}f$ from [Nievergelt 1999].

2 Problems

We begin with a list of problems, both serious gaps and trivial typographical errors, which we encountered in [Nievergelt 1999]. The mathematical errors are items 30 and 43 and, to a lesser extent, items 11, 25, 36 and 37. Items 30 and 43 are addressed in greater detail in subsequent sections of these notes. We worked from the first printing of [Nievergelt 1999]. After the writing of this section, we became aware that there is a second printing in which many of the typographical errors have been corrected. In the list below, the errors marked by an asterisk in the margin remain in the second printing.

1. page 10: Figure 1.7(c), vertical axis should be labeled $-4, 4$ instead of $5, 5$.

* 2. page 76: line 2 of displayed equation

$$= \frac{1-\sqrt{3}}{4} \cdot \frac{1-\sqrt{3}}{2} + h_2 \cdot 0 + h_1 \cdot 0 + h_0 \cdot 0$$

3. page 77: middle of the page

$$= -0 - 2\frac{\sqrt{3}}{8} + \frac{4-2\sqrt{3}}{8}$$

4. page 80: Example 3.6 (b)

$$4.5 = 9 \cdot 2^{-1} = 100.1_{\text{two}}$$

* 5. page 84: the lines

$$f(0)\varphi(r_0) \approx f(0)$$
$$f(1)\varphi(r_1) \approx 0$$
$$f(2)\varphi(r_2) \approx 0$$

are accurate but not precisely what is needed in the context. It would be better to say

$$\varphi(r_0) \approx 1$$
$$\varphi(r_1) = \varphi(r_0 + 1) \approx 0$$
$$\varphi(r_2) = \varphi(r_0 + 2) \approx 0$$

and because $0 < r_0 < 1$, we have

$$\varphi(r_k) = \varphi(r_0 + k) \approx 0$$

for $k \geq 3$ or $k \leq -1$, since then $r_0 + k > 3$ or $r_0 + k < 0$ (respectively). Thus, in the sum

$$\sum_{k=0}^{2^n-1} f(k)\varphi(r_0 + \ell - k),$$

the only nonzero term occurs when $\ell - k = 0$, that is, when $k = \ell$. Hence, for $0 \leq \ell \leq 2^n - 1$,

$$\sum_{k=0}^{2^n-1} f(k)\varphi(r_0 + \ell - k) \approx f(\ell)\varphi(r_0) \approx f(\ell).$$

6. page 86: line 5

$$(s_{-4},\ s_{-3},\ s_{-2},\ s_{-1};\ s_0,\ s_1,\ s_2,\ s_3;\ s_4,\ s_5,\ s_6,\ s_7)$$

7. page 86: for consistency, either line 9 should read

$$a_k = \sum_{r=k}^{k+3} \varphi(r - k)s_r$$

or line 18 (for example) should read

$$a_1 = s_2 \cdot \varphi(2 - 1) + s_3 \cdot \varphi(3 - 1)$$

(Since $\varphi(0) = \varphi(3) = 0$, this is merely a stylistic quibble.)

8. page 89: line 5

$$(s_{-4},\ s_{-3},\ s_{-2},\ s_{-1};\ s_0,\ s_1,\ s_2,\ s_3;\ s_4,\ s_5,\ s_6,\ s_7)$$

9. page 89: line 15

$$a_{-2} = s_{-2} \cdot \varphi(-2 + 2) + s_{-1} \cdot \varphi(-1 + 2) + s_0 \cdot \varphi(0 + 2) + s_1 \cdot \varphi(1 + 2)$$

10. page 90: 4th line from bottom, the last entry in the extension should be

$$s_{2^{n+1}-1}$$

* 11. page 97, top of page: it does not follow from ${}_D\Omega^T{}_D\Omega = 2I$ that ${}_D\Omega$ is invertible. Indeed, because ${}_D\Omega^T{}_D\Omega = 2I$, invertibility would imply ${}_D\Omega^T{}_D\Omega = {}_D\Omega{}_D\Omega^T = 2I$ which fails because the upper-left entry of ${}_D\Omega{}_D\Omega^T$ is $h_0^2 + h_3^2 < h_0^2 + h_1^2 + h_2^2 + h_3^2 = 2$. ${}_D\Omega$ would be invertible if it were extended as follows, where the box surrounds the $(0, 0)$ entry.

$$
{}_D\Omega = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & h_0 & h_3 & & & & & \cdots \\
\cdots & h_1 & -h_2 & & & & & \cdots \\
\cdots & h_2 & h_1 & \boxed{h_0} & h_3 & & & \cdots \\
\cdots & h_3 & -h_0 & h_1 & -h_2 & & & \cdots \\
\cdots & & & h_2 & h_1 & h_0 & h_3 & \cdots \\
\cdots & & & h_3 & -h_0 & h_1 & -h_2 & \cdots \\
\cdots & & & & & h_2 & h_1 & \cdots \\
\cdots & & & & & h_3 & -h_0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

This error reappears in the discussion of the inverse transform on page 103. The statement "By periodicity, the first two rows coincide with the rows with indices $2*(2^{n-1}-1)$ and $2*(2^{n-1}+1), \ldots$" is false with the given definition of ${}_D\Omega$. The formulas for a_0^{n-1} and a_1^{n-1} on lines 7 and 8 are fine.

12. page 98: line 4
$$a^{(2)}_{2^{n+1}-2}, a^{(2)}_{2^{n+1}-1}$$

13. page 101: line 4
$$+c_0^{(n-3)}\psi([r/8-1])$$

14. page 104: middle of page, first line of step 2
$$(c_0^{(0)}, c_1^{(0)})$$

15. page 105: top of page
$$a_1^{(1)} = h_3 a_1^{(0)} - h_0 c_1^{(0)} + h_1 a_0^{(0)} - h_2 c_0^{(0)}$$

16. page 105: line 7
$$a_2^{(1)} = h_2 a_0^{(0)} + h_1 c_0^{(0)} + h_0 a_1^{(0)} + h_3 c_1^{(0)}$$

∗ 17. page 156: top of page
$$\hat{\mathbf{f}} \mapsto \vec{\mathbf{f}} = {}_F^N\Omega\hat{\mathbf{f}}$$
(arrow replaced by hat over last symbol)

18. page 187: 3rd line after equation in middle of the page
$$(1.1787797-1)/2 \approx 0.09 = 9\%$$

∗ 19. page 190: line 4 of Proposition 6.25 should read
$$\text{for all real } t \in I$$

∗ 20. page 195: top of page
$$\begin{aligned}\textstyle\sum_{k=-N}^{N} |z_k \overline{w_k}| &= \langle (|z_{-N}|, \ldots, |z_0|, \ldots, |z_N|), (|w_{-N}|, \ldots, |w_0|, \ldots, |w_N|) \rangle \\ &\leq \|(|z_{-N}|, \ldots, |z_0|, \ldots, |z_N|)\|_2 \cdot \|(|w_{-N}|, \ldots, |w_0|, \ldots, |w_N|)\|_2\end{aligned}$$

∗ 21. page 196: line 4 of proof
$$= \frac{1}{2T}\left\{ f(s)e^{-iks\pi/T}\Big|_{-T}^{T} - \int_{-T}^{T} f(s)(-ik\pi/T)e^{-iks\pi/T} ds \right\}$$
(coefficient in front of integral deleted)

∗ 22. pages 197-198: 1/2T missing in various places, e.g. the equation in the statement of Corollary 6.41 should read
$$\sum_{k\in\mathbb{Z}} |c_{f,k}|^2 = \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 dt$$

23. page 199: line 2 of equation in proof

$$= \frac{1}{2T} \int_{-t-T}^{-t+T} f(x+t) \cdot \frac{\sin[N+\frac{1}{2}](x)\pi/T}{\sin(x)\pi/(2T)} dx$$

∗ 24. page 199: last line

$$= \frac{1}{2} \cdot \frac{1}{2T} \int_{-T}^{T} \frac{\sin[N+\frac{1}{2}](x)\pi/T}{\sin(x)\pi/(2T)} dx = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

∗ 25. page 209: in definition 7.8, the assumption that both functions are bounded is too restrictive because it does not cover the use of the convolution later in the chapter, e.g. Propositions 7.22, 7.31, definition of h in the proof of 7.34. It seems to suffice for the rest of the text to have convolutions of integrable functions one of which is bounded or both of which are in $\mathcal{L}^2$. The integrand in the definition is then either the product of a bounded function and an integrable function or the product of two $\mathcal{L}^2$ functions and hence is integrable. Of course, in results like 7.11, 7.31 it then is necessary to add suitable conditions on the functions as part of the hypothesis.

∗ 26. page 210: in the statement of Lemma 7.12, it seems to be relevant for the proof of Theorem 7.34 (see Section 4 of these notes) to note that the hypothesis that $\mathcal{F}f$ is integrable can be deleted. (The proof needs no changes for this.)

∗ 27. page 212: middle of page

$$\begin{aligned} f(r) &= \lim_{B\to\infty} (f * A_B)(r) \\ &= \lim_{B\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}f)(s) K_B(s) e^{i\cdot r\cdot s} ds \end{aligned}$$

∗ 28. page 217: line 6 of displayed equation

$$= \int_{[x-R,x+R]} |\{f(x)-f(t)\} w_c(x-t)| dt + \int_{\mathbb{R}\setminus[x-R,x+R]} |\{f(x)-f(t)\} w_c(x-t)| dt$$

∗ 29. page 222: line 3

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(z) \cdot e^{-i\cdot w\cdot(z+u)} dz = (\mathcal{F}f)(w) \cdot e^{-iuw}$$

(dx replaced by dz)

∗ 30. page 224, lines 5–8 of the proof of 7.34: why does the Fourier inversion theorem apply to $\mathcal{F}h$? There seems to be a missing proof that $\mathcal{F}h = |\mathcal{F}f|^2$ is integrable. See Section 4.

∗ 31. page 227: line 2 of displayed equation at top of page, replace $=$ by $\leq$.

∗ 32. page 228: middle of page

$$\begin{aligned} &= \left| \int_{|x|>R} g(x) \left(e^{-irx}-1\right) e^{-iwx} dx + \int_{-R}^{R} g(x) \left(e^{-irx}-1\right) e^{-iwx} dx \right| \\ &\leq \int_{|x|>R} |g(x)| \left|e^{-irx}-1\right| dx + \int_{-R}^{R} |g(x)| \left|e^{-irx}-1\right| dx \end{aligned}$$

∗ 33. page 244 line 5 and page 245 line 16:

$$\mathcal{F}(T^{\circ n}g)(t) = \prod_{\ell=1}^{n} \left(\frac{1}{2} \sum_{k=0}^{N} h_k e^{-ikt/2^\ell} \right) \cdot (\mathcal{F}g)(t/2^n).$$

(subscript under product is $\ell = 1$ instead of $\ell = 0$)

∗ 34. page 244 sentence after equation (8.7): is it the sequence $(\mathcal{F}(T^{\circ n}))$ which was intended here? If not, why is this true? See also item 43 below.

∗ 35. page 248: top of page

$$\hat{\varphi}(t) := \lim_{n\to\infty} \big(\mathcal{F}T^{\circ n}g\big)(t) = \prod_{\ell=1}^{\infty} \left(\frac{1}{2} \sum_{k=0}^{N} h_k e^{-ikt/2^\ell} \right) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g$$

(coefficient added in front of $\int_{\mathbb{R}} g$)

∗ 36. page 248, statement and proof of Corollary 8.9:

(a) The use of formula (8.9) in the first sentence of the proof assumes via the application of the Fourier transform that g is integrable. Hence something needs to be said about φ being integrable.

(b) The last line of the proof is confusing in the context of what has been presented so far because $\mathcal{F}^{-1}$ has only been applied to functions in $\mathcal{L}^1$ whereas $\mathcal{F}\varphi_1$, $\mathcal{F}\varphi_2$ have not been shown or assumed to lie in $\mathcal{L}^1$. Of course this is unnecessary because of the injective nature of the Fourier transform, but this has not been stated. It might be better to say that $\mathcal{F}\varphi_1 = \mathcal{F}\varphi_2$ gives that $\mathcal{F}(\varphi_1 - \varphi_2) = 0$ belongs to $\mathcal{L}^1$ and hence, by Theorem 7.30, $(\varphi_1 - \varphi_2)(t) = \mathcal{F}^{-1}(0)(t) = 0$ at any point t where φ_1 and φ_2 are both continuous, where $\mathcal{F}^{-1}$ means specifically the operator given by $(\mathcal{F}^{-1}f)(t) = (1/\sqrt{2\pi}) \int_{\mathbb{R}} f(s)e^{ist}\, ds$. This conclusion is weaker than the given one (which does not mention points of continuity).

(c) In the statement of Corollary 8.9, say "There exists at most one *continuous integrable* function. . .".

∗ 37. page 248, lines 10–11: why does the existence of $\hat{\varphi}$ ensure that φ is a Fourier transform? The decay condition introduced farther down on the same page ensures that $\hat{\varphi} \in \mathcal{L}^1 \cap \mathcal{L}^2$. The fact that $\hat{\varphi} \in \mathcal{L}^1$ ensures the existence of $\varphi = \mathcal{F}^{-1}\hat{\varphi}$ where, by definition,

$$(\mathcal{F}^{-1}\hat{\varphi})(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(s)e^{ist}\, ds.$$

The Plancherel theorem also ensures that $\varphi \in \mathcal{L}^2$. If we were working with the Fourier transform on $\mathcal{L}^2$, which has not been introduced here, then it would also follow that $\mathcal{F}\varphi = \hat{\varphi}$. In the present context, we need $\varphi \in \mathcal{L}^1$ before we can say that $\hat{\varphi}$ is a Fourier transform. We know from Proposition 7.38 that φ is continuous. If we show that φ has compact support then it follows that $\varphi \in \mathcal{L}^1$. This does not seem to follow easily from the results in the text. See Section 5 below, particularly Remark 5.3.

38. page 249: middle of page, for consistency it might be better to write

$$|\sin(x)| \leq 1$$

∗ 39. page 249, lines 13, 14: This statement has already been proven. We know $(\mathcal{F}g)(t/2^n)$ converges from Lemma 8.6, and the convergence of the infinite product was the conclusion of the argument on page 247, and therefore their product converges as well. What is about to be proven is that there exist positive constants C and s so that $|\hat{\varphi}(t)| \leq C/(1 + |t|)^{1+s}$, which by the argument on page 248 ensures that $\hat{\varphi} \in \mathcal{L}^1 \cap \mathcal{L}^2$.

∗ 40. page 250: line 7

$$\left|e^{-ir}-1\right| = \left|\int_0^r (-i)e^{-it}dt\right| \leq \int_0^{|r|} |-i|\left|e^{-it}\right|dt = \int_0^{|r|} 1dt = |r|.$$

(absolute value in second last integral moved)

∗ 41. page 251: line -6

$$1+e^{-it/2^\ell} = 2e^{-it/2^{\ell+1}}\frac{e^{it/2^{\ell+1}}+e^{-it/2^{\ell+1}}}{2} = 2e^{-it/2^{\ell+1}}\cos(t/2^\ell+1)$$

(2's added in front of expressions after both equality signs)

42. page 255: top of page

$$= \left(\sum_{k=0}^{N} h_k h_{k-2(q-p)}\right)\frac{1}{4}\int_{\mathbb{R}} |g(w)|^2\,dw$$

∗ 43. page 255, first three lines of the proof of 8.17 (see also the sentence after the bulleted list on page 240): this statement seems to be false. What was shown in the previous section is that the Fourier transforms of the terms of the sequence $(T^{\circ n}g)_{n=0}^{\infty}$, rather than the terms themselves, converge uniformly. See Section 5 below.

44. page 257: line 2

$$\varphi_k^{(m)}$$

45. page 257: middle of page

$$p = 2m-n+1$$

46. page 258: top of page

$$= \sum_{p_1}(-1)^{p_1}h_{1-p_1}\int_{\mathbb{R}}\varphi(2(2^m x-h)-p_1)\psi(2^n x-k)\,dx$$

∗ 47. page 281: isn't section 9.2.3 nearly identical to section 3.2.1?

∗ 48. page 285: what is the point of Part D?

3 Solutions to a few exercises

In this section we provide a few more details for some of the arguments in the text and give the solutions to some exercises whose results are needed for the proofs in subsequent sections below.

EXERCISE 7.1: Calculate $(\mathcal{F}h)(w)$ for the function h define by

$$h(x) = \begin{cases} 1-|x| & \text{if } -1\leq x\leq 1, \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
(\mathcal{F}h)(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-ixw}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{-ixw}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{1}(1-x)e^{-ixw}\,dx + \int_{-1}^{0}(1+x)e^{-ixw}\,dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{1}e^{-ixw}\,dx - \int_{0}^{1}xe^{-ixw}\,dx + \int_{-1}^{0}e^{-ixw}\,dx + \int_{-1}^{0}xe^{-ixw}\,dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\left.\frac{e^{-ixw}}{-iw}\right|_{0}^{1} - \left.\frac{xe^{-ixw}}{-iw}\right|_{0}^{1} + \left.\frac{e^{-ixw}}{(-iw)^2}\right|_{0}^{1} + \left.\frac{e^{-ixw}}{-iw}\right|_{-1}^{0} + \left.\frac{xe^{-ixw}}{-iw}\right|_{-1}^{0} - \left.\frac{e^{-ixw}}{(-iw)^2}\right|_{-1}^{0}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{ie^{-iw}}{w} - \frac{i}{w} - \frac{ie^{-iw}}{w} - \frac{e^{-iw}}{w^2} + \frac{1}{w^2} + \frac{i}{w} - \frac{ie^{iw}}{w} + \frac{ie^{iw}}{w} + \frac{1}{w^2} - \frac{e^{iw}}{w^2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{2-e^{-iw}-e^{iw}}{w^2}\right) \\
&= \frac{1}{\sqrt{2\pi}w^2}\left(2-(\cos(-w)+i\sin(-w))-(\cos(w)+i\sin(w))\right) \\
&= \frac{1}{\sqrt{2\pi}w^2}\big(2-\cos(w)+i\sin(w)-\cos(w)-i\sin(w)\big) \\
&= \frac{1}{\sqrt{2\pi}w^2}(2-2\cos(w)) = \frac{\sqrt{2}(1-\cos(w))}{\sqrt{\pi}w^2} \\
&= \frac{\sqrt{2}(1-\cos 2(w/2))}{\sqrt{\pi}w^2} = \frac{\sqrt{2}(1-(\cos^2(w/2)-\sin^2(w/2))}{\sqrt{\pi}w^2} \\
&= \frac{\sqrt{2}(2\sin^2(w/2))}{\sqrt{\pi}w^2} = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(w/2)}{w/2}\right)^2
\end{aligned}
$$

EXERCISE 8.6: Consider Daubechies' coefficients h_0, h_1, h_2, h_3.

(a) Verify that

$$
\begin{aligned}
h(z) &= (1/2)(h_0 + h_1 z + h_2 z^2 + h_3 z^3) \\
&= [(1/2)(1+z)]^2[(1-\sqrt{3})z + (1+\sqrt{3})]/2,
\end{aligned}
$$

so that $K = 2$ and $q(z) = [(1-\sqrt{3})z + (1+\sqrt{3})]/2$.

Solution:

$$
\begin{aligned}
&[(1/2)(1+z)]^2[(1-\sqrt{3})z+(1+\sqrt{3})]/2 \\
=\ & \frac{1}{8}((1+2z+z^2)(z-\sqrt{3}z+1+\sqrt{3})) \\
=\ & \frac{1}{8}(z-\sqrt{3}z+1+\sqrt{3}+2z^2-2\sqrt{3}z^2+2z+2\sqrt{3}z+z^3-\sqrt{3}z^3+z^2+\sqrt{3}z^2) \\
=\ & \frac{1}{8}(1+\sqrt{3}+3z+\sqrt{3}z+3z^2-\sqrt{3}z^2+z^3-\sqrt{3}z^3) \\
=\ & \frac{1}{8}((1+\sqrt{3})+z(3+\sqrt{3})+z^2(3-\sqrt{3})+z^3(1-\sqrt{3})) \\
=\ & \frac{1}{2}(h_0+zh_1+z^2h_2+z^3h_3)
\end{aligned}
$$

(b) Verify that if $|z|=1$, then $|q(z)| \le \sqrt{3}$, so that $Q=\sqrt{3}$.

Solution:

$$
\begin{aligned}
q(z) &= [(1-\sqrt{3})z+(1+\sqrt{3})]/2 \\
|q(z)| &= |[(1-\sqrt{3})z+(1+\sqrt{3})]/2| \\
&= |(1-\sqrt{3})z/2|+(1+\sqrt{3})/2 \\
&\le |(1-\sqrt{3})/2|\cdot|z|+(1+\sqrt{3})/2 \\
&= (\sqrt{3}-1)/2\cdot 1+(1+\sqrt{3})/2 \\
&= (\sqrt{3}-1+\sqrt{3}+1)/2 \\
&= \sqrt{3}
\end{aligned}
$$

therefore $\max(q(z))=\sqrt{3}$ when $z=e^{-it}$ and so $Q=\sqrt{3}$.

(c) Verify that $K-\log_2(Q)=2-\log_2(\sqrt{3})>1$.

Solution:

$$
\begin{aligned}
K-\log_2(Q)=2-\log_2(\sqrt{3}) &> 1 \\
-\log_2(\sqrt{3}) &> -1 \\
\log_2(\sqrt{3}) &< 1 \\
2^{\log_2(\sqrt{3})} &< 2^1 \\
\sqrt{3} &< 2
\end{aligned}
$$

Exercise: For the proof of Proposition 6.33, show that

$$\frac{1}{2T}\int_{-T}^{T}\frac{\sin[N+\frac{1}{2}](t-s)\pi/T}{\sin(t-s)\pi/(2T)}\,ds=1$$

By Lemma 6.13,

$$\frac{\sin[N+\frac{1}{2}](t-s)\pi/T}{\sin(t-s)\pi/(2T)}=\sum_{k=-N}^{N}e^{ik(t-s)\pi/T}$$

Therefore,

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\frac{\sin[N+\frac{1}{2}](t-s)\pi/T}{\sin(t-s)\pi/(2T)}\,ds &= \frac{1}{2T}\int_{-T}^{T}\sum_{k=-N}^{N} e^{ik(t-s)\pi/T}\,ds \\
&= \frac{1}{2T}\int_{-T}^{T}\left(1+\sum_{k=1}^{N}(e^{ik(t-s)\pi/T}+e^{-ik(t-s)\pi/T})\right)ds \\
&= \frac{1}{2T}\int_{-T}^{T}\left(1+\sum_{k=1}^{N}2\cos(k(t-s)\pi/T)\right)ds \\
&= 1+\frac{1}{2T}\sum_{k=1}^{N}\frac{2\sin(k(t-s)\pi/T)}{-k\pi/T}\Bigg|_{-T}^{T} \\
&= 1+\frac{1}{T}\sum_{k=1}^{N}\frac{\sin(k(t-T)\pi/T)-\sin(k(t+T)\pi/T)}{-k\pi/T} \\
&= 1
\end{aligned}$$

since $[k(t+T)\pi/T]-[k(t-T)\pi/T]=2k\pi$ and hence $\sin(k(t-T)\pi/T)=\sin(k(t+T)\pi/T)$.

Exercise: page 251, justify the claims in lines 11 and 12.

$$\begin{aligned}
&\le Q^{\ell_t}\cdot\exp(B|t/2^{\ell_t}|)\le Q^{\ell_t}\cdot\exp(B) \\
&\le Q^{1+\log_2|t|}\exp(B)=Q\cdot Q^{\log_2|t|}\exp(B) \\
&\le e^Q(e^{\log Q})^{\log_2|t|}e^B=e^Qe^Be^{(\log Q)\log_2|t|} \\
&\le C\cdot e^{(\log Q)\log_2|t|}=C\cdot e^{(\log Q)\log|t|/\log 2} \\
&= C\cdot(e^{\log|t|})^{\log Q/\log 2}=C\cdot|t|^{\log Q/\log 2}
\end{aligned}$$

Because $Q=\max\{|q(z)|:|z|=1\}\ge q(1)=1$ (see top of page 250), $\log Q\ge 0$. As well, $\log 2>0$. So $\log Q/\log 2\ge 0$ and hence the function $x\mapsto x^{\log Q/\log 2}$ is nondecreasing. Thus,

$$C\cdot|t|^{\log Q/\log 2}\le C\cdot(1+|t|)^{\log Q/\log 2}.$$

This gives

$$\left|\prod_{\ell=1}^{\infty}q(e^{-it/2^\ell})\right|\le C\cdot(1+|t|)^{\log Q/\log 2}$$

when $|t|>1$. Then the inequality holds for all t if C is suitably increased since $(1+|t|)^{\log Q/\log 2}\ge 1$.

4 Fourier transforms

The main point of this section is to fill what appears to be a gap in the proof of the Plancherel identity, Theorem 7.34. However, we prove more than is needed for this. The material in this section is drawn from [Rudin 1987], except that we prove only the special cases which make sense for the Riemann integral in the place of the Lebesgue integral and for piecewise continuous functions in the place of Lebesgue measurable functions. The proofs are adapted accordingly.

Proposition 4.1 *For any $1 \le p < \infty$, if $f \in \mathcal{L}^p$ and f is piecewise continuous, then there is a continuous function g with support in some interval $[-A, A]$, such that*

$$\|f - g\|_p < \varepsilon.$$

PROOF. By definition of piecewise continuous, there is a strictly increasing sequence $(a_n : n \in \mathbb{Z})$ such that $\lim_{n\to\infty} a_n = \infty, \lim_{n\to-\infty} a_n = -\infty$, f is continuous on (a_n, a_{n+1}) for each $n \in \mathbb{Z}$, and $\lim_{x\to a_{n}-} f(x)$ and $\lim_{x\to a_{n}+} f(x)$, both exist.

Define $\lim_{x\to a_{n}-} f(x) = \ell_n$ and $\lim_{x\to a_{n}+} f(x) = L_n$.

Since $\lim_{x\to a_{n}+} f(x) = L_n$, there is a $\delta^+ > 0$ such that for $x > a_n$

$$|x - a_n| = x - a_n < \delta^+ \Rightarrow |f(x) - L_n| < 1.$$

Since $\lim_{x\to a_{n}-} f(x) = \ell_n$ there is a $\delta^- > 0$ such that for $x < a_n$

$$|x - a_n| = a_n - x < \delta^- \Rightarrow |f(x) - \ell_n| < 1.$$

Choose an $R > 0$ big enough so that

$$\left| \int_{-R}^{R} |f|^p - \int_{-\infty}^{\infty} |f|^p \right| < \varepsilon^p/2.$$

Note that whenever $x < -R < R < y$, we have

$$\left| \int_{x}^{y} |f|^p - \int_{-\infty}^{\infty} |f|^p \right| < \varepsilon^p/2$$

Choose m large enough so that $a_{-m} < -R < R < a_m$ and define $\varepsilon_0 = \varepsilon^p/(2(2m+1))$. Choose x_n, y_n close enough to a_n such that, for $n = -m+1, \cdots, m-1$,

$$a_n - \delta^- < x_n < a_n < y_n < a_n + \delta^+$$

and

$$(y_n - x_n) < \frac{\varepsilon_0}{(|L_n - \ell_n| + 2)^p}.$$

For $n = -m$

$$a_{-m} - \delta^- < x_{-m} < a_{-m} < y_{-m} < a_{-m} + \delta^+$$

and

$$(y_{-m} - x_{-m}) < \frac{\varepsilon_0}{(2\max(|L_{-m}|, |\ell_{-m}|) + 2)^p}.$$

And for $n = m$

$$a_m - \delta^- < x_m < a_m < y_m < a_m + \delta^+$$

and

$$(y_m - x_m) < \frac{\varepsilon_0}{(2\max(|L_m|, |\ell_m|) + 2)^p}.$$

Let g be the continuous function defined by

$$g(x) = \begin{cases} f(x) & \text{if } a_{-m} \leq x \leq a_m,\ x \notin \bigcup_{n\in\mathbb{Z}}[x_n, y_n] \\ \dfrac{f(y_n)-f(x_n)}{y_n - x_n}(x - x_n) + f(x_n) & \text{if } x \in [x_n, y_n], n = -m+1, \cdots, m-1 \\ \dfrac{f(y_{-m})}{y_{-m} - x_{-m}}(x - x_{-m}) & \text{if } x \in [x_{-m}, y_{-m}] \\ \dfrac{f(x_m)}{y_m - x_m}(y_m - x) & \text{if } x \in [x_m, y_m] \\ 0 & \text{otherwise} \end{cases}$$

We have, for $x_n \leq x \leq y_n$, $n = -m+1, \cdots, m-1$,

$$\min(L_n - 1, \ell_n - 1) < \min(f(x_n), f(y_n)) \leq g(x) \leq \max(f(x_n), f(y_n)) < \max(L_n + 1, \ell_n + 1)$$

and, for the same values of x and n, by the choice of x_n and y_n we have

$$\ell_n - 1 < f(x) < \ell_n + 1 \text{ for } x_n \leq x < a_n \quad \text{and} \quad L_n - 1 < f(x) < L_n + 1 \text{ for } a_n < x \leq y_n$$

and hence

$$\min(L_n - 1, \ell_n - 1) < f(x) < \max(L_n + 1, \ell_n + 1)$$

for all $x \in [x_n, y_n]$, $x \neq a_n$. This gives, for all $x \in [x_n, y_n]$, $x \neq a_n$

$$|f(x) - g(x)| < \max(L_n + 1, \ell_n + 1) - \min(L_n - 1, \ell_n - 1) \leq |L_n - \ell_n| + 2.$$

Thus,

$$\int_{x_n}^{y_n} |f(x) - g(x)|^p \leq \int_{x_n}^{y_n} (|L_n - \ell_n| + 2)^p = (|L_n - \ell_n| + 2)^p (y_n - x_n) < \varepsilon_0$$

For $x_{-m} \leq x \leq y_{-m}$,

$$|g(x)| \leq |f(y_{-m})| \leq |f(y_{-m}) - L_{-m}| + |L_{-m}| < |L_{-m}| + 1,$$

and hence, using $|f(x)| < |\ell_{-m}| + 1$ for $x_{-m} \leq x < a_{-m}$ and $|f(x)| < |L_{-m}| + 1$ for $a_{-m} < x \leq y_{-m}$, we get that when $x_{-m} \leq x \leq y_{-m}$, $x \neq a_{-m}$,

$$|f(x) - g(x)| \leq |f(x)| + |g(x)| < \max(|\ell_{-m}|, |L_{-m}|) + 1 + |L_{-m}| + 1 \leq 2\max(|\ell_{-m}|, |L_{-m}|) + 2.$$

Consequently,

$$\int_{x_{-m}}^{y_{-m}} |f(x) - g(x)|^p \, dx \leq (2\max(|L_{-m}|, |\ell_{-m}|) + 2)^p (y_{-m} - x_{-m}) < \varepsilon_0.$$

Similarly, when $x_m \leq x \leq y_m$ we get

$$|g(x)| \leq |f(x_m)| \leq |f(x_m) - \ell_m| + |\ell_m| < |\ell_m| + 1,$$

and therefore, when $x \neq a_m$,

$$|f(x) - g(x)| \leq |f(x)| + |g(x)| < \max(|\ell_m|, |L_m|) + 1 + |\ell_m| + 1 \leq 2\max(|\ell_m|, |L_m|) + 2$$

which gives

$$\int_{x_m}^{y_m} |f(x) - g(x)|^p \, dx \le (2\max(|L_m|, |\ell_m|) + 2)^p (y_m - x_m) < \varepsilon_0.$$

Therefore,

$$\begin{aligned}\int |f-g|^p &= \int_{x_{-m}}^{y_{-m}} |f-g|^p + \sum_{n=-m+1}^{m-1} \int_{x_n}^{y_n} |f-g|^p + \int_{x_m}^{y_m} |f-g|^p + \int_{\mathbb{R}\setminus[x_{-m},y_m]} |f-g|^p \\ &< \varepsilon_0 + (2m-1)\varepsilon_0 + \varepsilon_0 + \varepsilon^p/2 = (2m+1)\varepsilon_0 + \varepsilon^p/2 = \varepsilon^p/2 + \varepsilon^p/2 = \varepsilon^p\end{aligned}$$

Hence,

$$\|f-g\|_p = \left(\int |f-g|^p\right)^{\frac{1}{p}} < (\varepsilon^p)^{\frac{1}{p}} = \varepsilon$$

□

Proposition 4.2 (Cf. [Rudin 1987, Theorem 9.5]) *For any piecewise continuous function f on $\mathbb{R}$ and every $y \in \mathbb{R}$, let f_y be the translate of f defined by*

$$f_y(x) = f(x-y), \qquad (x \in \mathbb{R})$$

If $1 \le p < \infty$ and if $f \in \mathcal{L}^p$, the mapping

$$y \to f_y$$

is a uniformly continuous mapping of $\mathbb{R}$ into $\mathcal{L}^p$.

Proof. Fix an $\varepsilon > 0$. Since $f \in \mathcal{L}^p$, there is a continuous function g whose support lies in a bounded interval $[-A, A]$, such that

$$\|f-g\|_p < \varepsilon/3.$$

By the uniform continuity of g, there exists a $\delta \in (0, A)$ such that

$$|s-t| < \delta \Rightarrow |g(s) - g(t)| < \frac{\varepsilon}{3(3A)^{1/p}}$$

Now suppose that $|s-t| < \delta$. We may also assume that $s \le t$, the case $t \le s$ being completely symmetric. We have $g(x-s) = 0$ unless $-A < x-s < A$ and $g(x-t) = 0$ unless $-A < x-t < A$. So $g(x-s) = g(x-t) = 0$ unless $-A+s < x < A+s$ or $-A+t < x < A+t$ and hence, unless $-A+s < x < A+t$. This gives

$$\|g_s - g_t\|_p^p = \int_{-\infty}^{\infty} |g(x-s) - g(x-t)|^p \, dx = \int_{s-A}^{t+A} |g(x-s) - g(x-t)|^p \, dx$$

Also, for any x, $|(x-s) - (x-t)| = |s-t| < \delta$, so

$$|g(x-s) - g(x-t)| < \frac{\varepsilon}{3(3A)^{1/p}}.$$

Thus,

$$\begin{aligned}\|g_s - g_t\|_p^p &= \int_{s-A}^{t+A} |g(x-s) - g(x-t)|^p \, dx \\ &< \frac{\varepsilon^p}{3^p(3A)} \cdot x \,\Big|_{s-A}^{t+A} = \frac{\varepsilon^p}{3^p(3A)} \cdot (2A + t - s) \\ &< \frac{\varepsilon^p(2A+\delta)}{3^p(3A)} < \frac{\varepsilon^p(3A)}{3^p(3A)} = \varepsilon^p/3^p\end{aligned}$$

So that,

$$\|g_s - g_t\|_p < \varepsilon/3$$

As well, for any $s \in \mathbb{R}$,

$$(f-g)_s(x) = (f-g)(x-s) = f(x-s) - g(x-s) = f_s(x) - g_s(x) = (f_s - g_s)(x)$$

and the change of variables $u = x - s$ gives

$$\|f_s\|_p^p = \int_{-\infty}^{\infty} |f_s|^p dx = \int_{-\infty}^{\infty} |f(x-s)|^p dx = \int_{-\infty}^{\infty} |f(u)|^p du = \|f\|_p^p.$$

Hence,

$$\begin{array}{rcl} \|f_s - f_t\|_p & = & \|f_s - g_s + g_s - g_t + g_t - f_t\|_p \\ & < & \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p \\ & = & \|(f-g)_s\|_p + \|g_s - g_t\|_p + \|(g-f)_t\|_p \\ & = & \|f-g\|_p + \|g_s - g_t\|_p + \|g - f\|_p \\ & < & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{array}$$

Corollary 4.3 (Cf. [Rudin 1987, Theorem 9.6]) *If $f \in \mathcal{L}^1$, then $\hat{f}$ is continuous and vanishes at infinity. Moreover, $\|\hat{f}\|_\infty \leq (1/\sqrt{2\pi})\|f\|_1$.*

Proof. The inequality is immediate from the definition of $\hat{f}$: for any $x \in \mathbb{R}$,

$$|\hat{f}(x)| = \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-itx}\, dt \right| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(t)|\, dt = \frac{1}{\sqrt{2\pi}} \|f\|_1.$$

Continuity of $\hat{f}$ was established in Proposition 7.38 on page 227. There remains to show that $\hat{f}$ vanishes at infinity. Since $e^{i\pi} = -1$, we have, for $x \neq 0$,

$$\hat{f}(x) = -\int_{\mathbb{R}} f(t) e^{-ix(t+\pi/x)}\, dt = -\int_{\mathbb{R}} f(t - \pi/x) e^{-ixt}\, dt.$$

Hence,

$$2\hat{f}(x) = \int_{\mathbb{R}} \{f(t) - f(t - \pi/x)\}\, e^{-ixt}\, dt$$

so that

$$2|\hat{f}(x)| \leq \|f - f_{\pi/x}\|_1 = \|f_0 - f_{\pi/x}\|_1$$

which tends to 0 at $x \to \pm\infty$ by Proposition 4.2. □

The next theorem is the special case of the monotone convergence theorem where all the functions are continuous. In our only application of this, in the proof of Theorem 4.5 below, the argument can be substantially simplified, avoiding in particular the use of compactness of closed bounded intervals. See Remark 4.6.

Theorem 4.4 *Let $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, and $f : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for every $x \in \mathbb{R}$,*

$$(*) \qquad 0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots$$

and $\lim_{n\to\infty} f_n(x) = f(x)$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

in the sense that if one side is finite then so is the other and they are equal.

Proof. For $n = 1, 2, \ldots$, let $A_n = \int_{-\infty}^{\infty} f_n(x)\,dx$ and let $A = \int_{-\infty}^{\infty} f(x)\,dx$. From $(*)$ we get by integrating that

$$0 \le A_1 \le A_2 \le \ldots \le A_n \le \ldots \le A.$$

The limit $\lim_{n\to\infty} A_n$ therefore exists as a member of $[0, \infty]$ and is $\le A$. It is a matter of showing that it equals A. For that it is enough to show that if we fix a number c such that $c < A$, then for some n we have $c < A_n \le A$. So fix a c such that $c < A$. By definition,

$$A = \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx,$$

so there is an $R > 0$ such that $c < A_R \le A$, where $A_R = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$. Now fix a partition

$$-R = x_0, x_1, \ldots, x_k = R$$

of the interval $[-R, R]$ such that the Riemann sum

$$S = \sum_{i=0}^{k-1} m_i(x_{i+1} - x_i),$$

where m_i is the minimum value of f on the interval $[x_i, x_{i+1}]$, satisfies $c < S \le A$.

Now choose numbers $m_i' < m_i$ such that $c < S' \le A$, where

$$S' = \sum_{i=0}^{m-1} m_i'(x_{i+1} - x_i).$$

It is enough for this that we take m_i' so close to m_i that $m_i - m_i' < (S - c)/(2R)$, for then we have

$$\begin{aligned} S - S' &= \sum_{i=0}^{m-1} (m_i - m_i')(x_{i+1} - x_i) \\ &< \frac{S-c}{2R} \sum_{i=0}^{m-1} (x_{i+1} - x_i) = \frac{S-c}{2R}((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_m - x_{m-1})) \\ &= \frac{S-c}{2R}(x_m - x_0) = \frac{S-c}{2R}(R - (-R)) = S - c, \end{aligned}$$

and hence $S' = (S' - S) + S > (c - S) + S = c$.

Claim. For some n, we have $m_i' < f_n(x)$ for all $i = 0, \ldots, k-1$ and all $x \in [x_i, x_{i+1}]$. Fix $i \in \{0, \ldots, k-1\}$. If for each n there is a point $a_n \in [x_i, x_{i+1}]$ such that $f_n(a_n) \le m_i'$, then take a subsequence $\{a_{n_\ell}\}_{\ell=1}^{\infty}$ which converges to a point $a \in [x_i, x_{i+1}]$. We have $\lim_{n\to\infty} f_n(a) = f(a)$ and $m_i' < m_i \le f(a)$, so for some N and all $n \ge N$ we have $m_i' < f_n(a)$. By continuity of $f_N(a)$, there is a $\delta > 0$ such that for all $x \in (a - \delta, a + \delta)$, $m_i' < f_N(x)$ and hence, by $(*)$, we have that for all $x \in (a - \delta, a + \delta)$ and all $n \ge N$, $m_i' < f_n(x)$. But now for any ℓ large enough so that $n_\ell \ge N$ and $a_{n_\ell} \in (a - \delta, a + \delta)$, we have

$$m_i' < f_{n_\ell}(a_{n_\ell}) \le m_i',$$

contradiction. Hence, for some n_i, we have $f_{n_i}(x) > m_i'$ for all $x \in [x_i, x_{i+1}]$. The largest number n among the numbers n_i, $i = 1, \ldots, k-1$ then satisfies the claim because of $(*)$. This establishes the claim.

Now use the claim to get n such that $m_i' < f_n(x)$ for all $i = 0, \dots, k-1$ and all $x \in [x_i, x_{i+1}]$. We have

$$\begin{aligned} A_n &= \int_{-\infty}^{\infty} f_n(x)\,dx \geq \int_{-R}^{R} f_n(x)\,dx \\ &= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} f_n(x)\,dx > \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} m_i'\,dx \\ &= \sum_{i=0}^{k-1} m_i'(x_{i+1} - x_i) = S' > c. \end{aligned}$$

□

The next result corresponds to Theorem 7.34 of [Nievergelt 1999].

Theorem 4.5 (Cf. [Rudin 1987, Theorem 9.13]) *If $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ is piecewise continuous, then $\hat{f} \in \mathcal{L}^2$ and*

$$\|\hat{f}\|_2 = \|f\|_2.$$

Proof. Fix a piecewise continuous function $f \in \mathcal{L}^1 \cap \mathcal{L}^2$. With the notation $f_-(x) = f(-x)$, let $h = f * [(\overline{f})_-]$. As shown in the proof of Theorem 7.34 on page 224, $\hat{h} = |\hat{f}|^2$. We have

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)\overline{f(-y)}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+y)\overline{f(y)}\,dy = \frac{1}{\sqrt{2\pi}} \langle f_{-x}, f \rangle.$$

Therefore,

$$\sqrt{2\pi}|h(x)| \leq \|f_{-x}\|_2 \|f\|_2 = \|f\|_2^2.$$

Hence h is bounded, and $h \in \mathcal{L}^1$ by Proposition 7.11 because $f \in \mathcal{L}^1$ and $(\overline{f})_- \in \mathcal{L}^1$. By Proposition 4.2 (with $p = 2$), h is uniformly continuous because we have

$$|h(x) - h(y)| = |\langle f_{-x}, f \rangle - \langle f_{-y}, f \rangle| = |\langle f_{-x} - f_{-y}, f \rangle| \leq \|f_{-x} - f_{-y}\|_2 \|f\|_2.$$

Since $h \in \mathcal{L}^1$, Lemma 7.12 (in which the hypothesis that the Fourier transform is integrable can be deleted as mentioned in Section 2) gives

$$(h * A_B)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_B(t)\hat{h}(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{|t|/B}\hat{h}(t)\,dt.$$

Therefore, because h is continuous, integrable and bounded, Lemma 7.13 gives

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} e^{-|t|/B}\hat{h}(t)\,dt = \lim_{B\to\infty} (h * A_B)(0) = h(0) = \|f\|_2^2$$

And the monotone convergence theorem (Theorem 4.4 above or Remark 4.6 below) gives

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} e^{-|t|/B}\hat{h}(t)\,dt = \int_{-\infty}^{\infty} \hat{h}(t)\,dt = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt = \|\hat{f}\|_2^2.$$

So $\|f\|_2^2 = \|\hat{f}\|_2^2$ and hence $\|f\|_2 = \|\hat{f}\|_2$. □

Remark 4.6 The monotone convergence theorem is easily established for the special case needed in the foregoing proof. As in the proof of Theorem 4.4, we start with a number $c < \int_{-\infty}^{\infty} \hat{h}(t)\,dt$ and get an $R > 0$ for which $c < \int_{-R}^{R} \hat{h}(t)\,dt$. Then note that we have $\int_{-R}^{R} e^{-|t|/B}\hat{h}(t)\,dt \leq \int_{-R}^{R} \hat{h}(t)\,dt$ and

$$0 \leq \int_{-R}^{R} \hat{h}(t)\,dt - \int_{-R}^{R} e^{-|t|/B}\hat{h}(t)\,dt = \int_{-R}^{R} (1 - e^{-|t|/B})\hat{h}(t)\,dt \leq (1 - e^{-R/B}) \int_{-R}^{R} \hat{h}(t)\,dt$$

which converges to 0 as $B \to \infty$. Hence, for all large enough values of B we have $c < \int_{-R}^{R} e^{-|t|/B}\hat{h}(t)\,dt \leq \int_{-\infty}^{\infty} e^{-|t|/B}\hat{h}(t)\,dt \leq \int_{-\infty}^{\infty} \hat{h}(t)\,dt$. This shows that $\lim_{B\to\infty} \int_{-\infty}^{\infty} e^{-|t|/B}\hat{h}(t)\,dt = \int_{-\infty}^{\infty} \hat{h}(t)\,dt$.

5 Wavelet transforms

The claim in sections 8.1 and 8.2 that the arguments in section 8.1 show that the sequence $T^{\circ n}g$, where g is the characteristic function of $[0, 1)$, converges uniformly to φ seems to be incorrect. In this section, we follow [Daubechies 1988] to complete the proof, except that our proof addresses only the special case of interest to us here. The undefined notation in what follows comes from section 8.1 of [Nievergelt 1999].

The next proposition establishes a useful property of the trigonometric polynomial $m_0(\xi) = h(e^{i\xi})$ from which it follows that $|m_0(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$. (Here is our first instance of undefined notation. The polynomial h is the one from Definition 8.7 of [Nievergelt 1999].)

Proposition 5.1 (Cf. [Daubechies 1988, pages 942–945]) *For all $\xi \in \mathbb{R}$, $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$.*

Proof. Define

$$\begin{aligned} \alpha(\xi) &= h_0 + h_2 e^{i\xi}, \\ \beta(\xi) &= h_1 + h_3 e^{i\xi}. \end{aligned}$$

We have, using $h_0 h_2 + h_1 h_3 = 0$,

$$\begin{aligned} |\alpha(\xi)|^2 + |\beta(\xi)|^2 &= |h_0 + h_2 e^{i\xi}|^2 + |h_1 + h_3 e^{i\xi}|^2 \\ &= (h_0 + h_2 e^{i\xi})\overline{(h_0 + h_2 e^{i\xi})} + (h_1 + h_3 e^{i\xi})\overline{(h_1 + h_3 e^{i\xi})} \\ &= (h_0^2 + h_2^2 + 2h_0 h_2 \cos\xi) + (h_1^2 + h_3^2 + 2h_1 h_3 \cos\xi) \\ &= h_0^2 + h_1^2 + h_2^2 + h_3^2 = 2 \end{aligned}$$

Also

$$\begin{aligned} m_0(\xi) = h(e^{i\xi}) &= (1/2)(h_0 + h_1 e^{i\xi} + h_2 e^{i2\xi} + h_3 e^{i3\xi}) \\ &= (1/2)(\alpha(2\xi) + e^{i\xi}\beta(2\xi)) \end{aligned}$$

and

$$\begin{aligned} m_0(\xi + \pi) &= (1/2)(\alpha(2\xi + 2\pi) + e^{i\xi}e^{i\pi}\beta(2\xi + 2\pi)) \\ &= (1/2)(\alpha(2\xi) - e^{i\xi}\beta(2\xi)) \end{aligned}$$

This gives

$$\begin{aligned} |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 &= (1/2)(\alpha(2\xi) + e^{i\xi}\beta(2\xi))\overline{(1/2)(\alpha(2\xi) + e^{i\xi}\beta(2\xi))} \\ &\quad + (1/2)(\alpha(2\xi) - e^{i\xi}\beta(2\xi))\overline{(1/2)(\alpha(2\xi) - e^{i\xi}\beta(2\xi))} \\ &= (1/4)(|\alpha(2\xi)|^2 + |\beta(2\xi)|^2 + e^{-i\xi}\alpha(2\xi)\overline{\beta(2\xi)} + e^{i\xi}\overline{\alpha(2\xi)}\beta(2\xi) \\ &\quad + |\alpha(2\xi)|^2 + |\beta(2\xi)|^2 - e^{-i\xi}\alpha(2\xi)\overline{\beta(2\xi)} - e^{i\xi}\overline{\alpha(2\xi)}\beta(2\xi)) \\ &= (1/4)(2 + 2) = 1 \end{aligned}$$

and this completes the proof. □

Proposition 5.2 (Cf. [Daubechies 1988, Proposition 3.3]) *The piecewise constant functions η_ℓ defined recursively by $\eta_0 = \chi_{[-1/2,1/2)}$ and for each integer $\ell \geq 0$, $\eta_{\ell+1} = T\eta_\ell$, i.e.,*

$$\eta_{\ell+1}(x) = \sum_{k=0}^{3} h_k \eta_\ell(2x - k),$$

converge uniformly to the continuous function η_∞ defined by $\eta_\infty = \mathcal{F}^{-1}\hat{\eta}_\infty$, where

$$\hat{\eta}_\infty(\xi) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\xi/2^j).$$

PROOF. From Chapter 8 of [Nievergelt 1999], specifically the discussion after the proof of Corollary 8.9, Lemma 8.12 and Exercise 8.6 (see Section 3), we know that $\hat{\eta}_\infty$ belongs to $\mathcal{L}^1 \cap \mathcal{L}^2$. In particular, $\eta_\infty = \mathcal{F}^{-1}\hat{\eta}_\infty$ is defined. As an intermediate step, we show that $\mu_\ell \to \eta_\infty$ uniformly, where the μ_ℓ are defined recursively as the η_ℓ but starting with the initial function μ_0 defined by

$$\mu_0(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

By exercise 7.1 (see Section 3), the Fourier transform of μ_0 is given by

$$\hat{\mu}_0(\xi) = (\mathcal{F}\mu_0)(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \xi/2}{\xi/2} \right)^2$$

and hence (see equation (8.7) page 244)

$$\hat{\mu}_\ell(\xi) = (\mathcal{F}\mu_\ell)(\xi) = \left[\prod_{j=1}^{\ell} m_0(2^{-j}\xi) \right] (\mathcal{F}\mu_0)(\xi/2^\ell) = \frac{1}{\sqrt{2\pi}} \left[\prod_{j=1}^{\ell} m_0(2^{-j}\xi) \right] \left[\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}} \right]^2$$

From Lemma 8.6 and section 8.1.3, we know that the sequence of functions $(\mathcal{F}\mu_\ell) = (\hat{\mu}_\ell)$ converges uniformly on compact sets to $\hat{\eta}_\infty$. Since the functions $\hat{\mu}_\ell$ are integrable and the functions μ_ℓ are continuous and integrable, it follows from Proposition 7.30 that $\mu_\ell = \mathcal{F}^{-1}\hat{\mu}_\ell$.

Remark 5.3 The notation $\hat{\eta}_\infty$ is somewhat abusive since, on the one hand, this function is defined before η_∞ and, on the other hand, even after defining $\eta_\infty = \mathcal{F}^{-1}\hat{\eta}_\infty$, we do not yet know that $\hat{\eta}_\infty$ is the Fourier transform (according to Definition 7.4 of [Nievergelt 1999]) of η_∞. We are lacking for this the knowledge that η_∞ is integrable. After we show that $\mu_\ell \to \eta_\infty$ uniformly, it will then follow that η_∞ is zero outside of $[0,3]$ since μ_ℓ is zero outside $[-1/2^\ell, 3-1/2^\ell]$.

[It follows inductively that $\mu_{\ell+1}$ will be zero except for values of x for which for some $k = 0,1,2,3$ we have $-1/2^\ell \le 2x - k \le 3 - 1/2^\ell$ and hence $-(1/2^{\ell+1}) \le (k/2) - (1/2^{\ell+1}) \le x \le (3/2) + (k/2) - (1/2^{\ell+1}) \le 3 - (1/2^{\ell+1})$.]

We already know from Proposition 7.38 that since η_∞ is an inverse Fourier transform, it is continuous. Together with the fact that η_∞ is zero outside $[0,3]$, this will show that η_∞ is integrable and hence $\mathcal{F}\eta_\infty$ is defined and $\mathcal{F}\eta_\infty = \mathcal{F}(\mathcal{F}^{-1}\hat{\eta}_\infty) = \hat{\eta}_\infty$ by Theorem 7.30 (which is easily seen to hold with $\mathcal{F}$ and $\mathcal{F}^{-1}$ interchanged).

For all $\delta > 0$ and for all $R > 0$, we can find ℓ_0 such that for all $\ell \ge \ell_0$,

$$\int_{|\xi| \le R} d\xi \, |\hat{\mu}_\ell(\xi) - \hat{\eta}_\infty(\xi)| \le \delta.$$

On the other hand, $\hat{\eta}_\infty \in \mathcal{L}^1$ as explained above. It follows that for all $\delta > 0$ there exists $R > 0$ such that

$$\int_{|\xi| \ge R} d\xi \, |\hat{\eta}_\infty(\xi)| \le \delta.$$

$\mathcal{L}^1$-convergence of $\hat{\mu}_\ell$ to $\hat{\eta}_\infty$ implies uniform convergence of μ_ℓ to η_∞ since any point $x \in \mathbb{R}$ is a point of continuity for both μ_ℓ and η_∞ and hence

$$\begin{aligned} |\mu_\ell(x) - \eta_\infty(x)| &= |(\mathcal{F}^{-1}\hat{\mu}_\ell)(x) - (\mathcal{F}^{-1}\hat{\eta}_\infty)(x)| \\ &= \left| \int_{\mathbb{R}} \hat{\mu}_\ell(\xi) e^{i\xi x} \, d\xi - \int_{\mathbb{R}} \hat{\eta}_\infty(\xi) e^{i\xi x} \, d\xi \right| \\ &\le \int_{\mathbb{R}} |\hat{\mu}_\ell(\xi) - \hat{\eta}_\infty(\xi)| \cdot |e^{i\xi x}| \, d\xi = \|\hat{\mu}_\ell - \eta_\infty\|_1 \end{aligned}$$

The $\mathcal{L}^1$ convergence will follow if we can prove that for all $\delta > 0$, there exist R and ℓ_0 large enough so that for all $\ell \geq \ell_0$,

$$\int_{|\xi|\geq R} d\xi \, |\hat{\mu}_\ell(\xi)| \leq \delta.$$

We need to evaluate the integral

$$\int_{|\xi|\geq R} d\xi \, |P_\ell(\xi)| \left| \frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}} \right|^2,$$

where $P_\ell(\xi) = \prod_{j=1}^{\ell} m_0(\xi/2^j)$. To do this, we split the integrand into two parts, namely $|\xi| \geq 2^\ell \pi$ and $R \leq |\xi| \leq 2^\ell \pi$. To evaluate these two parts, we shall use the following three properties of P_ℓ.

(i) $|P_\ell(\xi)| \leq 1$ (since $|m_0(\xi)| \leq 1$ by Proposition 5.1)

(ii) $|P_\ell(\xi)| \leq C \left| \dfrac{2^{-\ell}\sin(\xi/2)}{\sin(\xi/2^{\ell+1})} \right|^2 (1+|\xi|)^\beta,$

where $\beta = \log Q/\log 2$, Q as in Lemma 8.12 page 249.

(iii) P_ℓ is periodic with period $2^{\ell+1}\pi$.

To see that (ii) holds, first note that as in the calculations on pp. 248–252, we have

$$(1/2)(1+e^{-i\xi}) = e^{-i\xi/2}(e^{i\xi/2}+e^{-i\xi/2})/2 = e^{-i\xi/2}\cos(\xi/2)$$

and hence

$$\begin{aligned}
|P_\ell(\xi)| &= \prod_{j=1}^{\ell} |m_0(\xi/2^j)| \\
&\leq \prod_{j=1}^{\ell} |(1/2)(1+e^{-i\xi/2^j})|^2 |q(e^{-i\xi/2^j})| \\
&\leq \prod_{j=1}^{\ell} |\cos(\xi/2^{j+1})|^2 |q(e^{-i\xi/2^j})| \\
&= \prod_{j=1}^{\ell} \left| \frac{\sin(\xi/2^j)}{2\sin(\xi/2^{j+1})} \right|^2 |q(e^{-i\xi/2^j})| \\
&= \left| \frac{2^{-\ell}\sin(\xi/2)}{\sin(\xi/2^{\ell+1})} \right|^2 \prod_{j=1}^{\ell} |q(e^{-i\xi/2^j})|
\end{aligned}$$

The arguments on pages 250–252 for infinite products give bounds for finite products as well. We have

$$\sum_{j=1}^{\ell} |q(e^{-it/2^j}) - 1| \leq \sum_{j=1}^{\ell} \frac{|t|}{2^j} \leq |t|B$$

and hence, exactly as in the argument spanning the bottom of page 250 and the top of page 251 with ∞ replaced everywhere by ℓ, we get

$$\left| \prod_{j=1}^{\ell} q(e^{-it/2^j}) \right| \leq \exp(B|t|).$$

The argument in the middle of page 251, with the ∞'s replaced by the appropriate bounds, then shows that for $|t| > 1$ and some positive constant C,

$$\left|\prod_{j=1}^{\ell} q(e^{-it/2^j})\right| \leq C(1+|t|)^{\log Q/\log 2}.$$

This holds then for all $t \in \mathbb{R}$ if C is replaced by a suitably large constant. This proves (ii).

We concentrate first on $|\xi| \geq 2^\ell \pi$. Using the periodicity of P_ℓ, we find

$$\int_{|\xi|\geq 2^\ell\pi} d\xi\, |P_\ell(\xi)| \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}}\right|^2 \leq C \int_{|\xi|\leq 2^\ell\pi} d\xi\, |P_\ell(\xi)|\, |\sin(\xi/2^{\ell+1})|^2.$$

[Proof:

$$\begin{aligned}
\int_{|\xi|\geq 2^\ell\pi} d\xi\, |P_\ell(\xi)| \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}}\right|^2 &= \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \int_{2^\ell\pi+(k-1)2^{\ell+1}\pi}^{2^\ell\pi+k2^{\ell+1}\pi} d\xi\, |P_\ell(\xi)| \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}}\right|^2 \\
&= \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \int_{-2^\ell\pi=2^\ell\pi-2^{\ell+1}\pi}^{2^\ell\pi} du\, |P_\ell(u)| \left|\frac{\sin(u/2^{\ell+1}+k\pi)}{u/2^{\ell+1}+k\pi}\right|^2 \\
&\qquad\qquad [\text{set } u = \xi - k2^{\ell+1}\pi] \\
&= \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \int_{|\xi|\leq 2^\ell\pi} d\xi\, |P_\ell(\xi)| \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}+k\pi}\right|^2 \\
&\qquad\qquad [\text{since } \sin(\theta+k\pi) = \pm\sin(\theta)]
\end{aligned}$$

Then use that for $k \in \mathbb{Z}$, $k \neq 0$, and $|\xi| \leq 2^\ell\pi$, we have $|k| \geq 1$ and hence $|k\pi + \xi/2^{\ell+1}| \geq |k\pi| - |\xi|/2^{\ell+1} \geq |k|\pi - 2^\ell\pi/2^{\ell+1} = (|k| - \frac{1}{2})\pi \geq (|k| - \frac{1}{2}|k|)\pi = \frac{1}{2}|k|\pi$ which gives

$$\frac{1}{|k\pi + \xi/2^{\ell+1}|^2} \leq \frac{1}{(\frac{1}{2}|k|\pi)^2} = \frac{4}{\pi^2}\frac{1}{|k|^2}.$$

which yields the desired inequality with $C = 4\sum_{k\in\mathbb{Z},\, k\neq 0} 1/|k|^2$.]

Choose $\lambda = 2^{-\alpha\ell}$, with $\alpha \in (0,1)$ to be fixed later. Since $\alpha\ell$ is positive, we have $0 < \lambda < 1$. Then, using $|\sin(x)| \leq |x|$ and $|\sin(x)| \leq 1$, we get

$$\int_{|\xi|\leq 2^\ell\pi} d\xi\, |P_\ell(\xi)|\, |\sin(\xi/2^{\ell+1})|^2 \leq (\lambda^2/4) \int_{|\xi|\leq 2^\ell\lambda} d\xi\, |P_\ell(\xi)| + \int_{2^\ell\lambda\leq|\xi|\leq 2^\ell\pi} d\xi\, |P_\ell(\xi)| \tag{1}$$

Notice that $2^\ell\lambda = 2^\ell 2^{-\alpha\ell} = 2^{(1-\alpha)\ell} > 1$ since $1 - \alpha > 0$. Recall that $|\sin x| \geq 2|x|/\pi$ for $|x| \leq \pi/2$.

[The case $-\pi/2 \leq x \leq 0$ follows by applying to $-x$ the fact that $\sin x \geq 2x/\pi$ when $0 \leq x \leq \pi/2$, so it suffices to prove the latter. The function $f(x) = (\sin x) - 2x/\pi$ satisfies $f'(x) = (\cos x) - 2/\pi$. As x increases from 0 to $\pi/2$, $\cos x$ decreases from 1 to 0, passing through $2/\pi$ at precisely one value of x, say $x = a$. This means that f is strictly increasing as x increases from 0 to a and is strictly decreasing as x increases from a to $\pi/2$. Since $f(0) = f(\pi/2) = 0$, it follows that f is nonnegative on $[0, \pi/2]$.]

Now using (i) and (ii) above, and the fact that for $|\xi| \le 2^\ell \lambda$ we have $|\xi/2^{\ell+1}| \le \lambda/2 < 1/2 < \pi/2$, we get

$$\begin{aligned}
\int_{|\xi|\le 2^\ell\lambda} d\xi\, |P_\ell(\xi)| &\le \int_{|\xi|\le 1} d\xi\, |P_\ell(\xi)| + C\int_{1\le|\xi|\le 2^\ell\lambda} d\xi\,(1+|\xi|)^\beta \frac{1}{|2^\ell \sin(\xi/2^{\ell+1})|^2} \\
&\le \int_{-1}^{1} 1\, d\xi + C\int_{1\le|\xi|\le 2^\ell\lambda} d\xi\,(1+|\xi|)^\beta \frac{1}{2^{2\ell}\cdot(2|\xi/2^{\ell+1}|/\pi)^2} \\
&= 2 + C\pi^2 \int_{1\le|\xi|\le 2^\ell\lambda} d\xi\,(1+|\xi|)^\beta \frac{1}{|\xi|^2} \\
&\le 2 + 2C\pi^2 \int_1^\infty dx\, \frac{(1+x)^\beta}{x^2} = C_1
\end{aligned}$$

where C_1 is finite because

$$\int_1^\infty \frac{(1+x)^\beta}{x^2}\,dx \le \int_1^\infty \frac{(2x)^\beta}{x^2}\,dx = \int_1^\infty \frac{2^\beta}{x^{2-\beta}}\,dx$$

and $2-\beta > 1$ by exercise 8.6(c).

On the other hand,

$$\begin{aligned}
\int_{2^\ell\lambda\le|\xi|\le 2^\ell\pi} d\xi\, |P_\ell(\xi)| &\le \int_{2^\ell\lambda\le|\xi|\le 2^\ell\pi} d\xi\, C\left|\frac{2^{-\ell}\sin(\xi/2)}{\sin(\xi/2^{\ell+1})}\right|^2 (1+|\xi|)^\beta \\
&\le C2^{-2\ell}(1+2^\ell\pi)^\beta \int_{2^\ell\lambda\le|\xi|\le 2^\ell\pi} d\xi\, \frac{1}{\sin^2(\xi/2^{\ell+1})} \\
&\le C2^{-2\ell}(1+2^\ell\pi)^\beta 2^\ell\cdot 2\int_\lambda^\pi du\, \frac{1}{\sin^2(u/2)} \qquad\qquad [u=\xi/2^\ell] \\
&\le 2C2^{-\ell}(1+2^\ell\pi)^\beta \int_\lambda^\pi du\, \frac{1}{(2(u/2)/\pi)^2} \\
&\le 2\pi^2 C2^{-\ell}(2^\ell+2^\ell\pi)^\beta \frac{1}{\lambda^2}(\pi-\lambda) \\
&\le C_2 2^{\ell(\beta-1)}(1/\lambda^2)
\end{aligned}$$

Putting it all together, and choosing $\alpha = (1-\beta)/4 \in (0,1)$, this implies that (1) is

$$\le\ (1/4)\lambda^2 C_1 + C_2 2^{\ell(\beta-1)}(1/\lambda^2)\ =\ (1/4)2^{-2\alpha\ell}C_1 + C_2 2^{-4\alpha\ell}2^{\alpha\ell}\ \le\ C_3 2^{-2\alpha\ell}\ =\ C_3 2^{-2\ell(1-\beta)/4}. \tag{2}$$

This clearly tends to zero for $\ell \to \infty$.

We now evaluate the integral of $|\hat{\mu}_\ell|$ over $R \le |\xi| \le 2^\ell\pi$. Using (ii) we find

$$\begin{aligned}
\int_{R\le|\xi|\le 2^\ell\pi} d\xi\, |P_\ell(\xi)| \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}}\right|^2 &\le C\int_{R\le|\xi|\le 2^\ell\pi} d\xi \left|\frac{\sin(\xi/2)}{2^\ell\sin(\xi/2^{\ell+1})}\right|^2 (1+|\xi|)^\beta \left|\frac{\sin(\xi/2^{\ell+1})}{\xi/2^{\ell+1}}\right|^2 \\
&= 4C\int_{R\le|\xi|\le 2^\ell\pi} d\xi\,(1+|\xi|)^\beta|\xi|^{-2}\sin^2(\xi/2) \\
&\le 8C\int_R^\infty dx\, \frac{(1+x)^\beta}{x^2}
\end{aligned}$$

Since $2-\beta > 1$, this tends to zero for $R \to \infty$, uniformly in ℓ. Together with (2), this proves that

$$\int_{|\xi|\ge R} d\xi\, |\hat{\mu}_\ell(\xi)|$$

can be made as small as wanted by choosing ℓ and R large enough. As pointed out above, this proves $\|\hat{\mu}_\ell - \hat{\eta}_\infty\|_1 \to 0$ as $\ell \to \infty$.

Finally, we only need to show that uniform convergence of the μ_ℓ implies uniform convergence of the η_ℓ. The two functions μ_0 and η_0 agree on integers,

$$\mu_0(0) = \eta_0(0) = 1,$$

$$\mu_0(k) = \eta_0(k) = 0 \text{ for } k \in \mathbb{Z},\ k \neq 0.$$

Using the recursion relation which both the μ_ℓ and the η_ℓ satisfy, one sees that this implies, for all $\ell = 0, 1, 2, \ldots$,

$$\eta_\ell(k/2^\ell) = \mu_\ell(k/2^\ell) \text{ for all } k \in \mathbb{Z}.$$

Since η_∞ is an inverse Fourier transform, it is uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| \leq \delta \Rightarrow |\eta_\infty(x) - \eta_\infty(y)| \leq \varepsilon/2.$$

There also exist ℓ_0 such that, for all $\ell \geq \ell_0$ and all y,

$$|\eta_\infty(y) - \mu_\ell(y)| \leq \varepsilon/2.$$

Fix any $\ell \geq \ell_0$ such that $2^{-\ell} \leq \delta$. Let $x \in \mathbb{R}$. It follows inductively that η_ℓ is constant on each interval of the form $[(-2^{-\ell-1}) + 2^{-\ell}k, -2^{-\ell-1} + 2^{-\ell}(k+1))$, $k \in \mathbb{Z}$.

[η_0 is constant on each interval $[-(1/2) + k, -(1/2) + k + 1)$ and if η_ℓ is constant on each interval of the form

$$(*) \qquad \left[\frac{-1}{2^{\ell+1}} + \frac{k}{2^\ell},\ \frac{-1}{2^{\ell+1}} + \frac{k+1}{2^\ell}\right)$$

then for $j = 0, 1, 2, 3$ the transformation $x \mapsto 2x - j$ carries the interval

$$(**) \qquad \left[\frac{-1}{2^{\ell+2}} + \frac{k}{2^{\ell+1}},\ \frac{-1}{2^{\ell+2}} + \frac{k+1}{2^{\ell+1}}\right),$$

into an interval of the form $(*)$ (with k replaced by $k - 2^{\ell+1}j$) and hence $\eta_{\ell+1} = T\eta_\ell$ is constant on each interval $(**)$.]

Choose $k \in \mathbb{Z}$ such that $x \in [(-2^{-\ell-1}) + 2^{-\ell}k, -2^{-\ell-1} + 2^{-\ell}(k+1))$. This gives

$$-2^{-\ell-1} \leq x - 2^{-\ell}k < -2^{-\ell-1} + 2^{-\ell} = 2^{-\ell-1}$$

and hence

$$|x - 2^{-\ell}k| \leq 2^{-\ell-1} < \delta.$$

Since $2^{-\ell}k$ also belongs to the interval $[(-2^{-\ell-1}) + 2^{-\ell}k, -2^{-\ell-1} + 2^{-\ell}(k+1))$, we have $\eta_\ell(x) = \eta_\ell(2^{-\ell}k) = \mu_\ell(2^{-\ell}k)$, and we get

$$|\eta_\ell(x) - \eta_\infty(x)| \leq |\mu_\ell(2^{-\ell}k) - \eta_\infty(2^{-\ell}k)| + |\eta_\infty(2^{-\ell}k) - \eta_\infty(x)| \leq \varepsilon.$$

Since ε was arbitrary, this shows that η_ℓ converges uniformly to η_∞ for $\ell \to \infty$. $\square$

Suppose g and h are functions related by $g(x) = h(x-a)$. Then

$$\begin{aligned}
(Tg)(x) &= \sum_{k=0}^{3} h_k g(2x-k) \\
&= \sum_{k=0}^{3} h_k h(2x-k-a) \\
&= \sum_{k=0}^{3} h_k h(2(x-a/2)-k) \\
&= (Th)(x-a/2)
\end{aligned}$$

It follows by induction that

$$(T^{\circ n} g)(x) = (T^{\circ n} h)(x - a/2^n)$$

If the sequence $(T^{\circ n}h)$ converges uniformly to a uniformly continuous function φ, then so does $(T^{\circ n}g)$. To see this, fix $\varepsilon > 0$. Choose $\delta > 0$ so that $|x-y| < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon/2$. Choose n_0 large enough so that for any $n \geq n_0$, $|a/2^n| < \delta$ and all $x \in \mathbb{R}$, $|(T^{\circ n}h)(x) - \varphi(x)| < \varepsilon/2$, then for any $n \geq n_0$ and any $x \in \mathbb{R}$ we have

$$\begin{aligned}
|(T^{\circ n}g)(x) - \varphi(x)| &= |(T^{\circ n}h)(x - a/2^n) - \varphi(x)| \\
&\leq |(T^{\circ n}h)(x - a/2^n) - \varphi(x - a/2^n)| + |\varphi(x - a/2^n) - \varphi(x)| \\
&\leq \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

Since $\chi_{[0,1)}(x) = \chi_{[-1/2,1/2)}(x - 1/2)$, we get first part of the following result.

Corollary 5.4 *The sequence $(T^{\circ n}g)$, where $g = \chi_{[0,1)}$, converges uniformly to a continuous function φ supported by $[0,3]$ and satisfying*

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(\xi/2^j).$$

The function φ satisfies the recurrence relation $T\varphi = \varphi$.

Proof. For the last statement, from $(T^{\circ(n+1)}g)(x) = \sum_{k=0}^{3} h_k (T^{\circ n}g)(2x-k)$ get by taking $\lim_{n\to\infty}$ of both sides that $\varphi(x) = \sum_{k=0}^{3} h_k \varphi(2x-k) = (T\varphi)(x)$. $\square$

6 Computations

We present here the results of some computational experimentation based on examples and exercises in [Nievergelt 1999]. The computations were carried out using a hybrid of C programs and Maple commands. The C programs were used to generate files containing strings which could be pasted into maple commands. This somewhat awkward, but nevertheless quite efficient, way of proceeding enabled us in the short time at our disposition to generate plots without learning how to program in Maple or how to generate plots in C. (The first author already knew how to program in C and the second author already knew how to generate plots in Maple.) We used the program in [Press et al. 1993] for applying the wavelet transform and its inverse.

The program `daubechies.c` listed in Appendix A generates values of the Daubechies scaling function φ. The values are calculated on the dyadic rationals using the basic recursion relation satisfied by φ. The program has two variants. One variant (corresponding to the lines marked `"version 2"` in the code) stores the values to a file `phi.dat` using `fwrite`. The file `phi.dat` then becomes a table from which the values can be retrieved using `fread`. The other variant (corresponding to the lines marked `"version 1"` in the code) writes a string of blocks of the form "`[`a`,`b`], `" to a file `phi.txt`, where a is a decimal representation, to three digits after the decimal, of a dyadic rational and b is a decimal representation, of the same type, of the value of φ at the dyadic rational in question. The values of a run over the dyadic rationals with denominator $2^8 = 256$ in the interval $[0, 3]$. The string in the file `phi.txt` resulting from running `"version 1"` of the program was pasted into a suitable Maple plot command. The resulting plot is also given in Appendix A.

The program `wt.c` listed in Appendix B generates the wavelet transform of an array of numbers whose length is a power of 2. It is a minor variation on the one in [Press et al. 1993]. It was applied to the sequence of 64 hourly temperatures (in °C) at Charlottetown, from August 15, 2004 00:00 to August 17, 2004 15:00 inclusive, obtained from the site

`http://www.climate.weatheroffice.ec.gc.ca/climateData/canada_e.html`

We made a first step toward compressing the data by zeroing out entries in the transform whose absolute value was smaller than the value of the variable `tolerance`. The degree of the compression achieved, or more precisely just the proportion of zeros in the resulting transform, is indicated in Appendix C for two values of `tolerance`. As an indication of the quality of the data obtained from the inverse transform of the compressed transform, we also give in Appendix C graphs of the original function and the inverse transforms corresponding to the same two values of `tolerance`.

Finally, Appendix D contains the Maple code for working out Exercise 8.5 page 252. We offer it here without further commentary.

References

[Daubechies 1988] I. Daubechies, *Orthonormal Bases of Compactly Supported Wavelets*, Comm. Pure Appl. Math., 41 (1988) 909–996.

[Hewitt and Stromberg 1965] E. Hewitt, K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965.

[Nievergelt 1999] Y. Nievergelt, *Wavelets Made Easy*, Birkhäuser, Boston, 1999.

[Press et al. 1993] W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes in C*, 2nd ed., Cambridge University Press, Cambridge, UK, 1993. Also available online via `http://www.nr.com/` The specific section we used is at `http://www.library.cornell.edu/nr/bookcpdf/c13-10.pdf`

[Rudin 1987] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, Inc., New York, NY, 1987.

Appendix A

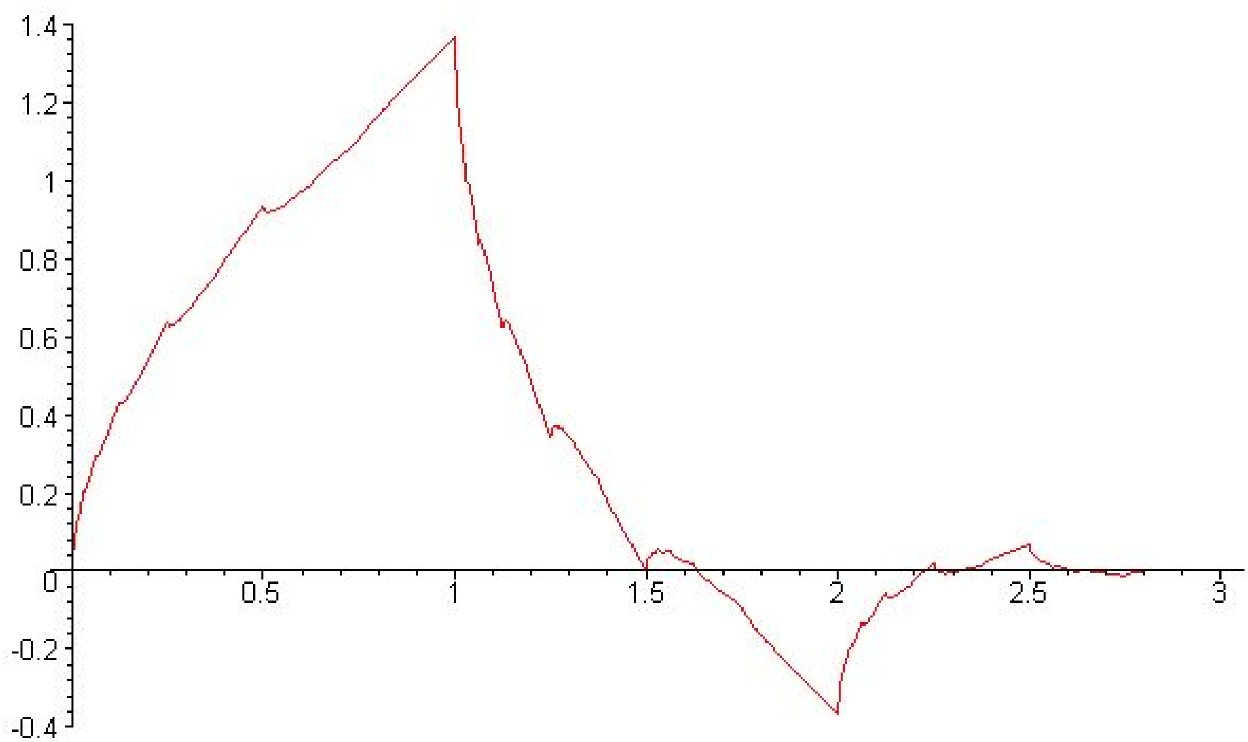
```
//This is the program daubechies.c
#include <stdio.h>
#include <stdlib.h>

#define H0 0.683012701892219
#define H1 1.18301270189222
#define H2 0.316987298107781
#define H3 -0.183012701892219

#define p0 0
#define p1 1.36602540378444
#define p2 -0.366025403784439
#define p3 0

double calculate_phi(int i, double *phi1, int power);

void main()
{
    int n, N, m, k, i, pow, power;
    double ans, *phi1, *phi2, x, y;
    FILE *daubPtr;

    //"version 1"
/*
    if ((daubPtr = fopen("phi.txt", "w")) == NULL)
        printf("File could not be opened\n");
*/
//  "version 2"
    if ((daubPtr = fopen("phi.dat", "wb")) == NULL)
        printf("File could not be opened\n");

    n = 8;
    pow = 1;
    for (i=0; i<n; i++) pow *= 2;
    N = 3*pow + 1;

    phi1 = (double*)malloc(N*sizeof(double));
    phi2 = (double*)malloc(N*sizeof(double));
    phi1[0] = p0;
    phi1[1] = p1;
    phi1[2] = p2;
    phi1[3] = p3;
    power = 1;
    for(k=1; k<=n; k++)
    {
        power *= 2;
        m = 3*power+1;
        for(i=0; i<m; i++) phi2[i] = calculate_phi(i, phi1, power);
```

```
        for(i=0; i<m; i++) phi1[i] = phi2[i];
    }


    //  "version 1"
/*
    for(i=0; i<m; i++) fprintf(daubPtr, "[%7.3f, %7.3f],", i/(float)power , phi2[i]);
*/
    //  "version 2"
    fwrite(phi2, 1, m*sizeof(double), daubPtr);


    //  "version 2"
    //  Close the file and reopen to retrieve a particular
    //  value of phi.  For "version 1", comment out the remaining
    //  lines down to the line before the return statement.
    fclose(daubPtr);
    if ((daubPtr = fopen("phi.dat", "rb")) == NULL)
        printf("File could not be opened\n");

    x = 1.5;

    //  find i giving the closest dyadic rational to x of the form i/power
    i=0;
    while(i <= x * power)i++;
    if (i - (x*power) > 0.5) i--;

    //  read the appropriate entry from phi.dat
    fseek(daubPtr, i*sizeof(double), SEEK_SET);
    fread(&y, 1, sizeof(double), daubPtr);
    printf("phi(%f)=%f\n", x,y);

    return;
}

double calculate_phi(int i, double *phi1, int power)
{
    double sum=0;
    int k, j, power0;
    power0 = power/2;
    j = -1;
    for(k=0; k*power0 <= i && k<4; k++) j++;

    sum += H0*phi1[i];
    if(j>=1) sum += H1*phi1[i-power0];
    if(j>=2) sum += H2*phi1[i-2*power0];
    if(j>=3) sum += H3*phi1[i-3*power0];

    return sum;
}
```

The Maple commands

```
> daub:=[[  0.000,   0.000],[  0.004,   0.065],[  0.008,
  0.095],              {...}                   [  2.984,
  -0.000],[  2.988,   0.000],[  2.992,   0.000],[  2.996,
  -0.000],[  3.000,   0.000]]:
> with(plots):
Warning, the name changecoords has been redefined

> pointplot(daub, style=line, color=red);
```

where `{...}` stands for a long list, obtained by running `daubechies.c`, of pairs with first coordinates going from 0 to 256 by increments of $1/256 = 0.00390625 \approx 0.004$, produce the following plot.



Appendix B

```
//This is the program wt.c
#include <stdio.h>
#include <stdlib.h>

#define C0 0.4829629131445341
#define C1 0.8365163037378079
#define C2 0.2241438680420134
#define C3 -0.1294095225512604

#define N 64

void wt1(double *a, unsigned long n, int isign);
void daub4(double * a, unsigned long n, int isign);

int main()
{
    int i;

    double tolerance = 0.5;

    double *data;
    FILE *daubPtr;

    //Hourly temperature, Charlottetown PE,
    //from August 15, 2004 00:00 to August 17, 2004 15:00 inclusive

    double inputdata[N]={19.2, 18.6, 18.2, 17.4, 17.4, 17.4, 17.1, 18.4,
                         19.8, 20.9, 22.1, 23.0, 23.7, 21.4, 21.7, 20.7,
                         19.9, 19.2, 18.8, 17.8, 17.3, 17.2, 17.6, 16.7,
                         17.1, 17.2, 16.2, 15.4, 16.3, 15.3, 15.6, 16.9,
                         18.1, 19.4, 20.5, 21.9, 21.9, 21.4, 21.0, 20.5,
                         20.8, 20.4, 19.5, 17.9, 17.7, 17.4, 16.1, 17.1,
                         16.7, 15.3, 15.3, 15.1, 15.2, 14.6, 14.2, 15.0,
                         14.6, 14.4, 15.1, 17.0, 17.1, 18.1, 19.5, 19.6};

    if ((daubPtr = fopen("output.txt", "w")) == NULL)
        printf("File could not be opened\n");

    data = (double *) malloc(N * sizeof(double));
    for (i=0;i<N;i++) data[i] = inputdata[i];

    for (i=0;i<N;i++) printf("%6.2f",data[i]);
    printf("\n\n");

    for (i=0;i<N;i++) fprintf(daubPtr, "[%d,%6.2f], ",i, data[i]);

    wt1(data,N,1);
```

```
    for (i=0;i<N;i++) printf("%6.2f",data[i]);
    printf("\n\n");

    for (i=0;i<N;i++) if (-tolerance < data[i] && data[i]<tolerance) data[i]=0;

    printf("data after compression using tolerance = %f:\n",tolerance);

    for (i=0;i<N;i++) printf("%6.2f",data[i]);
    printf("\n\n");

    wt1(data,N,-1);

    for (i=0;i<N;i++) printf("%6.2f",data[i]);
    printf("\n");

    for (i=0;i<N;i++) fprintf(daubPtr, "[%d,%6.2f], ",i, data[i]);

    return 0;
}

/*******************************************************************/
/*                                                                 */
/*   One-dimensional discrete wavelet transform. This routine,     */
/*   a slightly modified version of an algorithm in Numerical      */
/*   Recipes in C, implements the pyramid algorithm, replacing     */
/*   a[0..n-1] by its wavelet transform (for isign=1), or          */
/*   performing the inverse operation (for isign=-1).              */
/*                                                                 */
/*   Note that n MUST be an integer power of 2.                    */
/*                                                                 */
/*******************************************************************/

void wt1(double *a, unsigned long n, int isign)
{
    int i;
    unsigned long nn;
    if (n < 4) return;
    if (isign >= 0) for (nn=n;nn>=4;nn>>=1) daub4(a,nn,isign);
                    //Wavelet transform. Start at largest hierarchy,
                    //and work towards smallest.
    else for (nn=4;nn<=n;nn<<=1) daub4(a,nn,isign);
           //Inverse wavelet transform. Start at smallest hierarchy,
           //and work towards largest.
}

/* Applies the Daubechies 4-coefficient wavelet filter to data
vector a[0,...,n-1] (for isign=1) or applies its transpose (for
isign=-1). Used hierarchically by wt1. */
void daub4(double * a, unsigned long n, int isign)
{
```

```
    double *wksp;
    unsigned long nh,nh1,i,j;
    if (n < 4) return;
    wksp = (double *) malloc(n * sizeof(double));
    nh = n >> 1;      //nh is n divided by 2
    if (isign == 1) {
    //Apply filter.
        for (i=0,j=0;j<=n-4;j+=2,i++) {
            wksp[i]    = C0*a[j] + C1*a[j+1] + C2*a[j+2] + C3*a[j+3];
            wksp[i+nh] = C3*a[j] - C2*a[j+1] + C1*a[j+2] - C0*a[j+3];
        }
        wksp[i]    = C0*a[n-2] + C1*a[n-1] + C2*a[0] + C3*a[1];
        wksp[i+nh] = C3*a[n-2] - C2*a[n-1] + C1*a[0] - C0*a[1];
    }
    if (isign == -1) {
    //Apply transpose filter.
        wksp[0] = C2*a[nh-1] + C1*a[n-1] + C0*a[0] + C3*a[nh];
        wksp[1] = C3*a[nh-1] - C0*a[n-1] + C1*a[0] - C2*a[nh];
        for (i=0,j=2;i<nh-1;i++) {
            wksp[j++] = C2*a[i] + C1*a[nh+i] + C0*a[i+1] + C3*a[nh+i+1];
            wksp[j++] = C3*a[i] - C0*a[nh+i] + C1*a[i+1] - C2*a[nh+i+1];
        }
    }
    for (i=0;i<n;i++) a[i]=wksp[i];
    free(wksp);
}
```

Appendix C

Wavelet coefficients output by program `wt.c`, rounded to 2 decimals. (The output has been reformatted.)

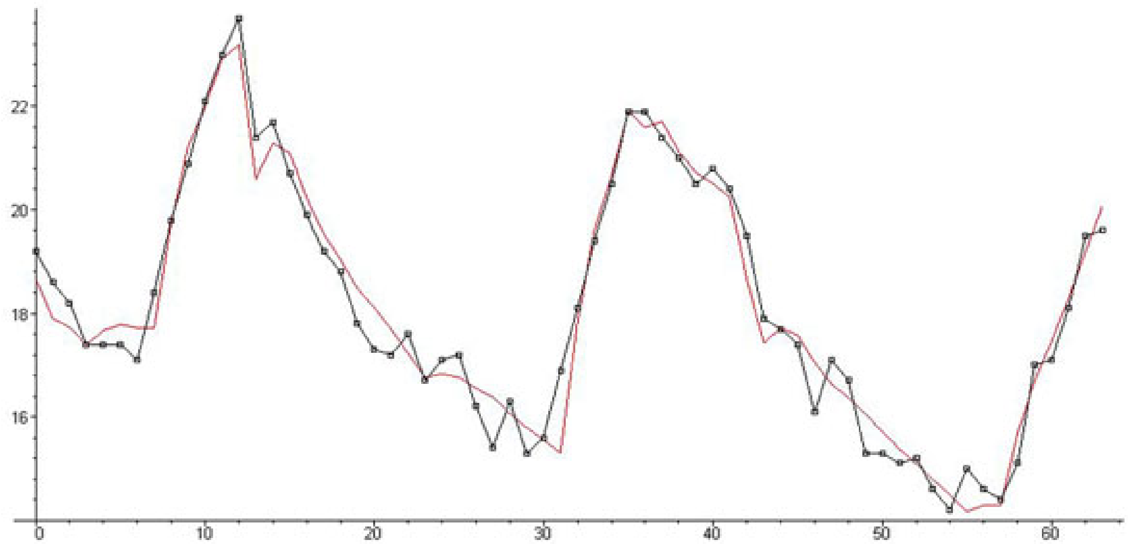
```
105.51 99.54  3.60 10.08 -0.95  8.88 -8.84 -3.29
  3.79 -1.00 -0.97  2.72  0.78 -0.86 -2.87  0.46
 -0.89 -0.25  0.97 -0.05 -0.42  0.94 -0.99 -0.35
  1.12  1.27 -0.65 -0.23 -0.38 -1.50 -0.06  0.75
  0.17 -0.10 -0.73  0.13  0.13  1.47  0.29 -0.07
  0.25 -0.26  0.56 -0.02  0.05  0.70 -0.65 -0.04
 -0.12  0.42  0.04  0.23  0.40 -0.13 -0.98  0.66
 -0.08  0.30 -0.61  0.06 -0.70 -0.20  0.58  0.16
```

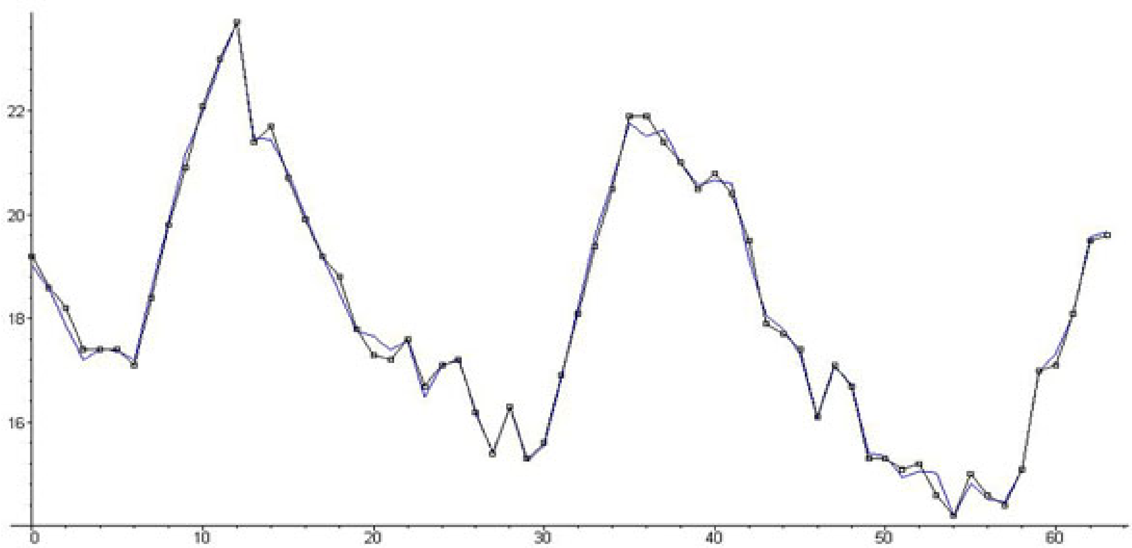
With `tolerance = 1`, this becomes the following sequence of coefficients, in which 50 out of 64 are zero. The inverse transform produces the red line (the one without boxes if you are looking at this in black and white) on the first graph on the next page. The graph with the small boxes represents the original data set, with one box for each data point. On each vertical line through one of these boxes there is a data point on the red line as well. (The corresponding boxes are omitted.)

```
105.51 99.54  3.60 10.08  0.00  8.88 -8.84 -3.29
  3.79  0.00  0.00  2.72  0.00  0.00 -2.87  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
  1.12  1.27  0.00  0.00  0.00 -1.50  0.00  0.00
  0.00  0.00  0.00  0.00  0.00  1.47  0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
  0.00  0.00  0.00  0.00  0.00  0.00  0.00  0.00
```

With `tolerance = 0.5`, this becomes the following sequence of coefficients, in which 30 out of 64 are zero. The inverse transform produces the blue line on the second graph on the next page.

```
105.51 99.54  3.60 10.08 -0.95  8.88 -8.84 -3.29
  3.79 -1.00 -0.97  2.72  0.78 -0.86 -2.87  0.00
 -0.89  0.00  0.97  0.00  0.00  0.94 -0.99  0.00
  1.12  1.27 -0.65  0.00  0.00 -1.50  0.00  0.75
  0.00  0.00 -0.73  0.00  0.00  1.47  0.00  0.00
  0.00  0.00  0.56  0.00  0.00  0.70 -0.65  0.00
  0.00  0.00  0.00  0.00  0.00  0.00 -0.98  0.66
  0.00  0.00 -0.61  0.00 -0.70  0.00  0.58  0.00
```


22
20
18
16
0
10
20
30
40
50
60


22
20
18
16
0
10
20
30
40
50
60

Appendix D

The following Maple code is for solving exercise 8.5 page 252 and more generally for experimenting with plots of the graphs of the terms of the sequence $(T^{\circ n} g)$ for various choices of g.

```
> with(plots):
Warning, the name changecoords has been redefined
>
> H[0]:=(1+3^(1/2))/4:
> H[1]:=(3+3^(1/2))/4:
> H[2]:=(3-3^(1/2))/4:
> H[3]:=(1-3^(1/2))/4:
>
> g:=x->piecewise(x<0, 0, 0<=x and x<=1, 1, 1<x, 0):
> g:=x->piecewise(x<-1/2, 0, -1/2<=x and x<1/2, 1, 1/2<=x, 0):
> g:=x->piecewise(x<-1, 0, -1<=x and x<1, 1-abs(x), 1<=x, 0):
> g:=x->piecewise(x<0, 0, 0<=x and x<2, 1-abs(x-1), 2<=x, 0):
> g:=x->piecewise(x<0, 0, 0<=x and x<3/2, 16*((3/4)-abs(x-3/4))/9, 3/2<=x, 0):
>
> Tg :=x->(H[0]*g(2*x)   + H[1]*g(2*x-1)   + H[2]*g(2*x-2)   + H[3]*g(2*x-3)):
> T2g:=x->(H[0]*Tg(2*x)  + H[1]*Tg(2*x-1)  + H[2]*Tg(2*x-2)  + H[3]*Tg(2*x-3)):
> T3g:=x->(H[0]*T2g(2*x) + H[1]*T2g(2*x-1) + H[2]*T2g(2*x-2) + H[3]*T2g(2*x-3)):
> T4g:=x->(H[0]*T3g(2*x) + H[1]*T3g(2*x-1) + H[2]*T3g(2*x-2) + H[3]*T3g(2*x-3)):
> T5g:=x->(H[0]*T4g(2*x) + H[1]*T4g(2*x-1) + H[2]*T4g(2*x-2) + H[3]*T4g(2*x-3)):
>
> a0:=plot(g(x), x=-1..3):
> a1:=plot(Tg(x), x=-1..3):
> a2:=plot(T2g(x), x=-1..3):
> a3:=plot(T3g(x), x=-1..3):
> a4:=plot(T4g(x), x=-1..3):
> a5:=plot(T5g(x), x=-1..3, color=black):
> display(a3,a5);
```

Running the code in the following order

(1) the first block and then the second block defining `H[0]`, `H[1]`, `H[2]`, `H[3]`

(2) the fourth definition of `g`

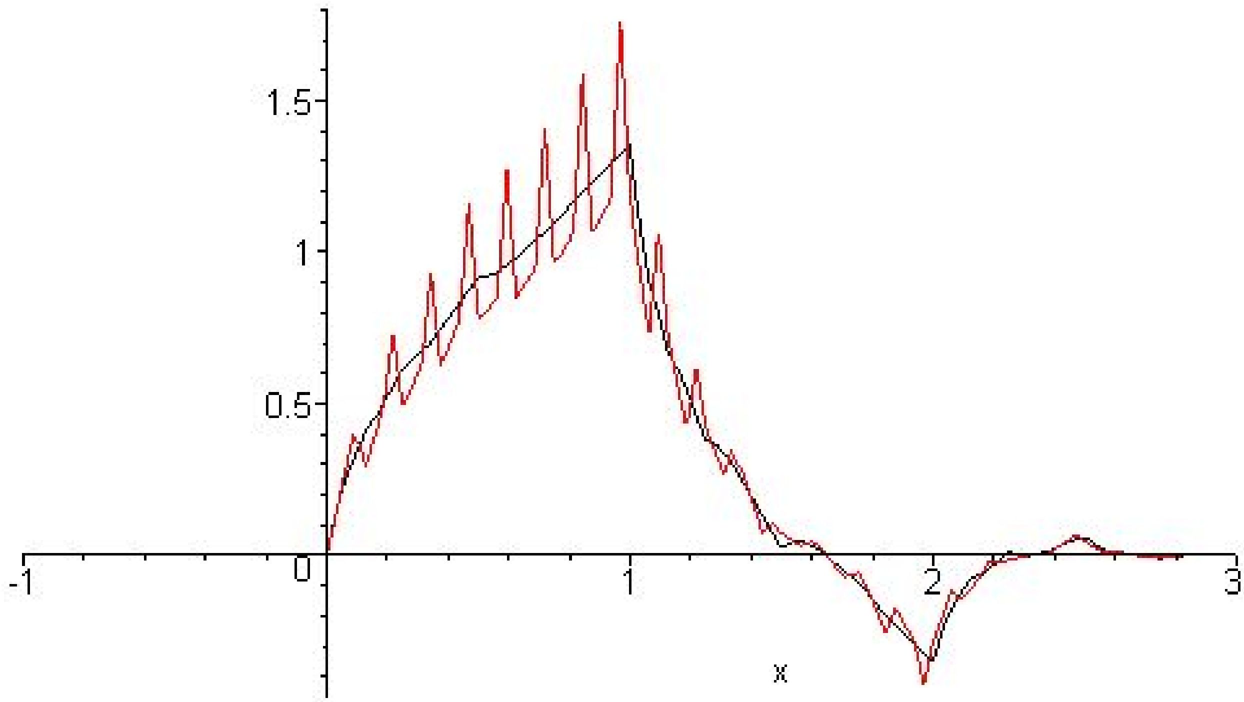
(3) the fourth block up to the definition of `T5g` inclusive

(4) the line defining `a5`

(5) the fifth definition of `g` followed by the fourth block of code up to the definition of `T3g` inclusive and then the line defining `a3`

(6) the last line of code

produces the following plot. (The black line is the smoother of the two graphs. The other is in Maple's default red.)


1.5
1
0.5
-1
0
1
2
3
x